C24 Dynamical Systems

Lecture 7: Bifurcations

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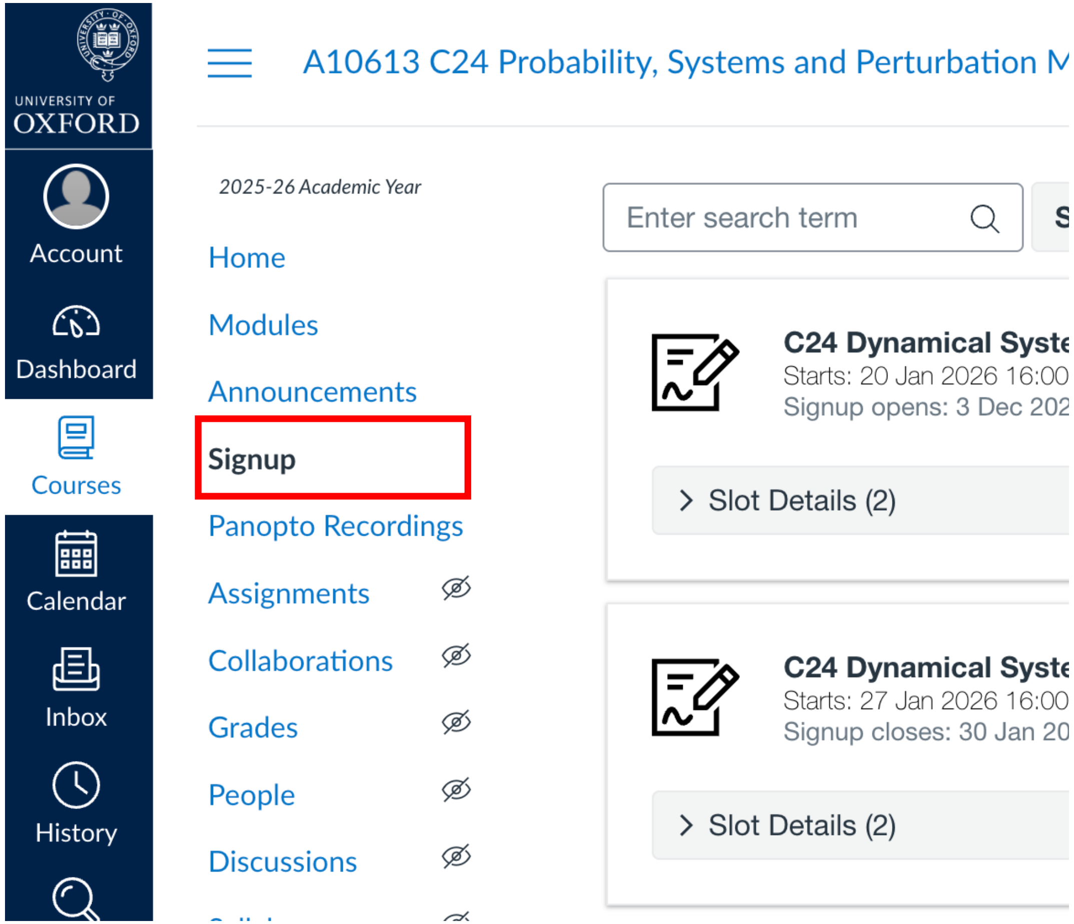
C24 Dynamical Systems Classes

- **Class 1**
  Hilary Term week 1
  Tuesday 20 Jan 16:00-18:00 LR3
  Friday 23 Jan 09:00-11:00 LR3

- **Class 2**
  Hilary Term week 2
  Tuesday 27 Jan 16:00-18:00 LR3
  Friday 30 Jan 09:00-11:00 LR3



Lecture 7 overview

- So far, we've considered autonomous systems with a single set of parameters, which were assumed to be known
- Now we will focus on how system behaviour changes depending on the values of the **constant parameters** of the system model
- Equilibrium points can change positions and character as the parameters change, leading to a **bifurcation** in the response
- This lecture will focus on categorizing bifurcations, and on providing criteria that can be used to classify them

Local bifurcations

- Until now our focus has been autonomous systems of the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

- Recall (Lecture 1) that we can also consider $\mathbf{f}$ to depend on a constant vector of parameters $\mathbf{p} \in \mathbb{R}^p$:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}; \mathbf{p})$$

- This lecture considers the structural stability of solution topology in phase space near equilibrium points as a function of the vector $\mathbf{p}$

- Here $\mathbf{p}$ is called a **bifurcation vector** or **bifurcation parameter** because the character of solutions may branch (bifurcate) if the parameter values change

1-D bifurcations

- The simplest systems to consider are autonomous systems with solutions on the (1-D) phase line with a single parameter:

$$\dot{x} = f(x; p), \qquad x, p \in \mathbb{R}$$

- A bifurcation occurs when the number or type of equilibrium points changes as parameter p is changed, e.g. stable to unstable

- Three types of 1-D bifurcation:
  - **saddle-node**
  - **transcritical**
  - **pitchfork**

- Bifurcations are analyzed using "normal forms" – standardized equations representing various classes of problem

Saddle-node bifurcation

- The normal form of a system with a saddle-node bifurcation is

$$\dot{x} = p - x^2$$

- There are stationary points when $0 = p - x^2 \;\; \Rightarrow \;\; x = \pm\sqrt{p}$
  - $p > 0 \Rightarrow$ two equilibria (one unstable, one stable)

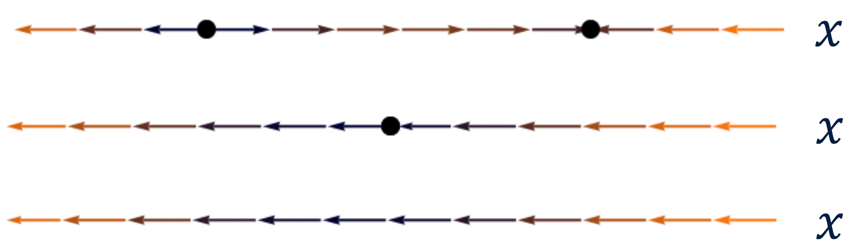
  - $p = 0 \Rightarrow$ one equilibrium point (a saddle)

  - $p < 0 \Rightarrow$ no equilibrium points

- A **bifurcation diagram** shows positions and types of equilibria (vertical axis) as p varies (horizontal axis). Solid lines: stable equilibria, dashed lines: unstable equilibria

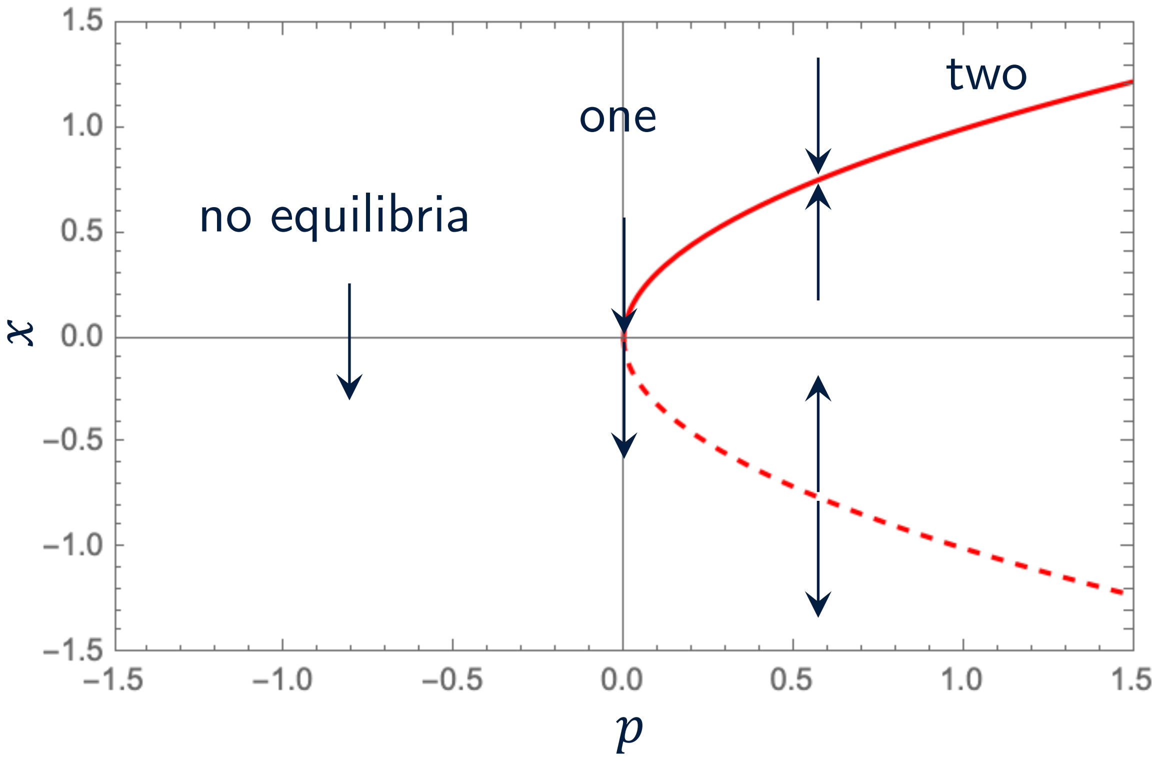
Saddle-node bifurcation diagram

- Normal form $\dot{x} = p - x^2$
  - $p > 0$
  - $p = 0$
  - $p < 0$



characteristic flows
on the phase line

- Bifurcation diagram:



Transcritical bifurcation

- The normal form of a system with a transcritical bifurcation is

$$\dot{x} = px - x^2 = x(p - x)$$

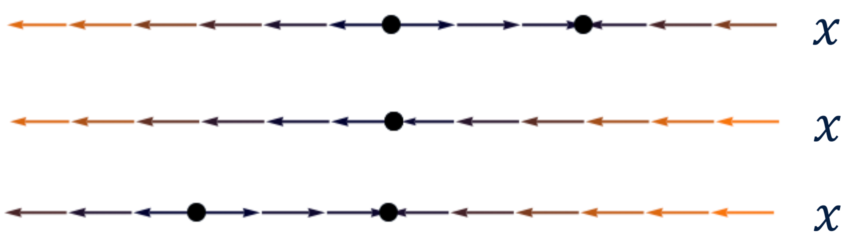
- There are equilibrium points at $x = 0$ and $x = p$
  - $p > 0 \Rightarrow$ two equilibria (one unstable, one stable)

  - $p = 0 \Rightarrow$ one equilibrium point (a saddle)

  - $p < 0 \Rightarrow$ two equilibria (one unstable, one stable)

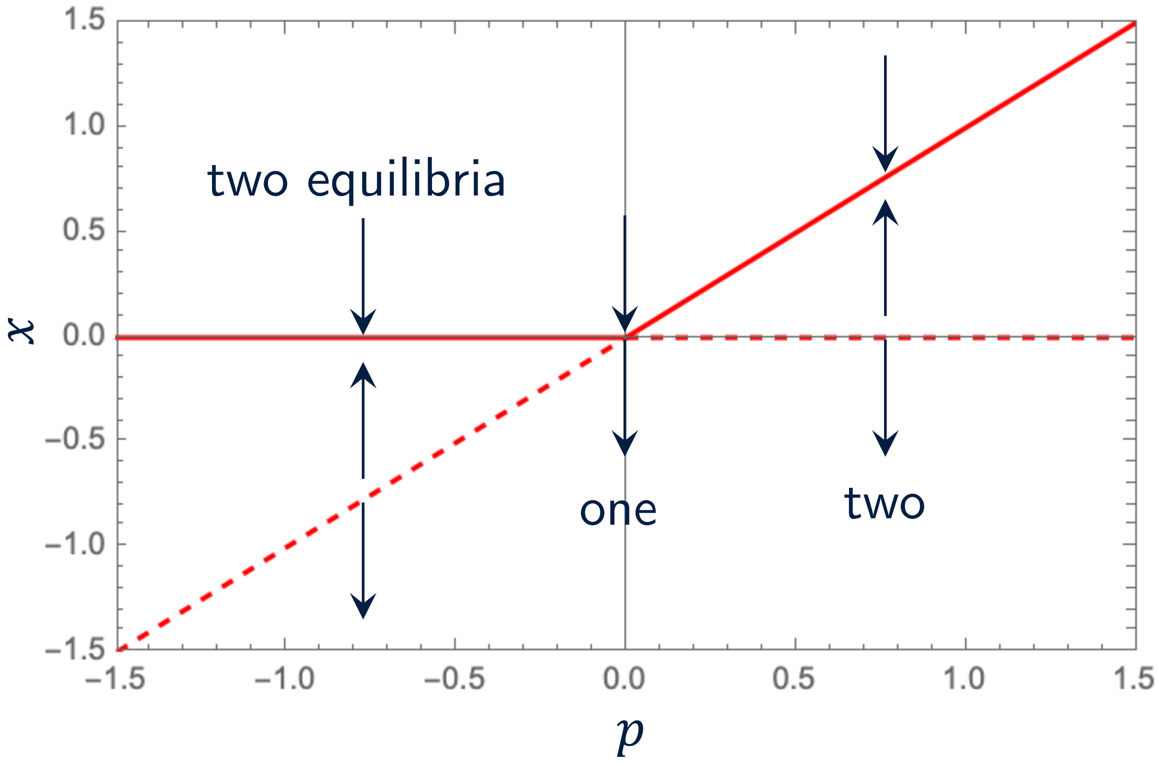
- There is always a stationary point at $x = 0$, but its stability depends on p: the equilibria switch types as p passes through the saddle point at $p = 0$

Transcritical bifurcation diagram

- Normal form $\dot{x} = px - x^2$
  - $p > 0$ 
  - $p = 0$
  - $p < 0$

characteristic flows on the phase line

- Bifurcation diagram:



Pitchfork bifurcation

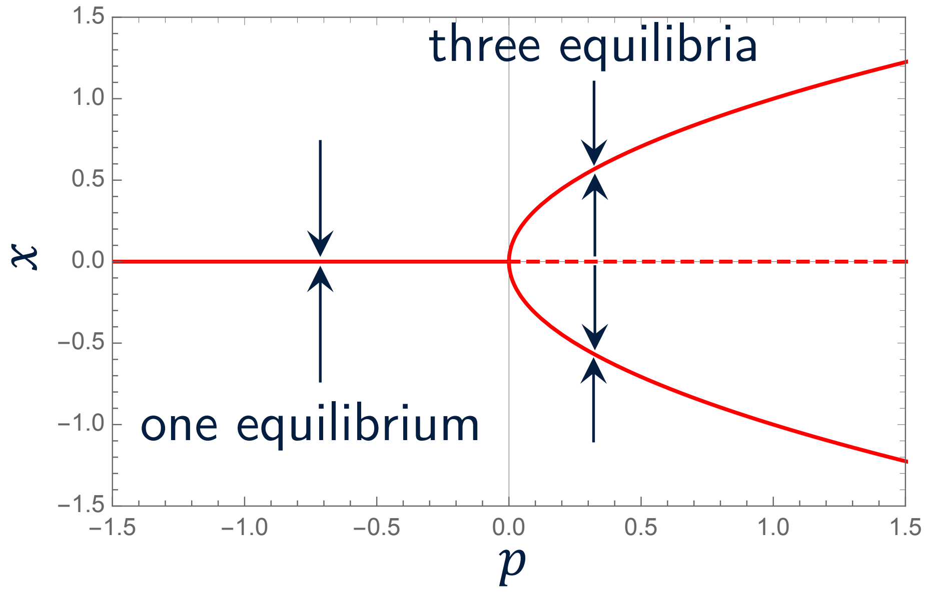
- The normal form of a system with a pitchfork bifurcation is

$$\dot{x} = px - x^3 = x(\sqrt{p} + x)(\sqrt{p} - x)$$

- There are stationary points at $x = 0$ and, if $p > 0$ at $x = \pm\sqrt{p}$
  - $p > 0 \Rightarrow$ three equilibria (one unstable, two stable)

  - $p \leq 0 \Rightarrow$ one equilibrium (stable)

- Bifurcation diagram:



Tangency conditions

- For a one-dimensional autonomous system, the locations x_0, p_0 of bifurcation points are identified by tangency conditions

$$f(x_0, p_0) = 0 \qquad \left.\frac{\partial f}{\partial x}\right|_{x_0, p_0} = 0$$

- The first condition says that x_0 is an equilibrium point; the second says that x_0 is a root with multiplicity two, hence a non-hyperbolic equilibrium

- For example consider the transcritical bifurcation:

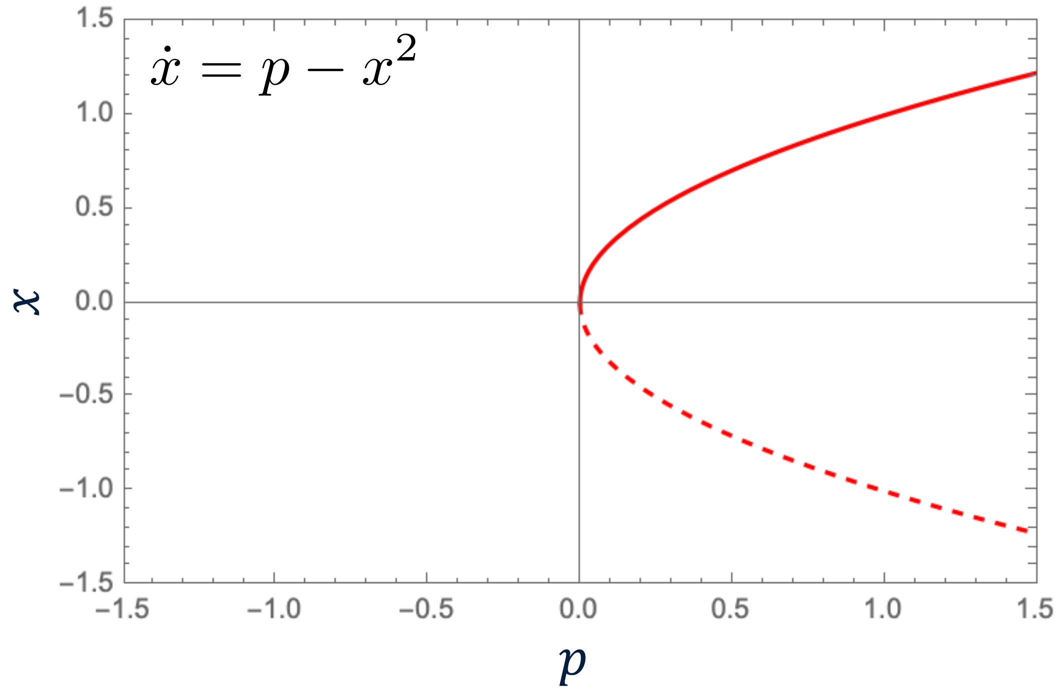
$$\begin{aligned} (px - x^2)|_{p_0, x_0} &= x_0(p_0 - x_0) = 0 \\ \frac{\partial}{\partial x}(px - x^2)|_{p_0, x_0} &= p_0 - 2x_0 = 0 \end{aligned} \Rightarrow (x_0, p_0) = (0, 0)$$

- These are the general conditions that determine whether a bifurcation exists; additional conditions classify the bifurcation

Tangency for saddle-node bifurcation

- For a bifurcation we need

$$f(x_0, p_0) = 0$$
$$\left.\frac{\partial f}{\partial x}\right|_{x_0, p_0} = 0$$



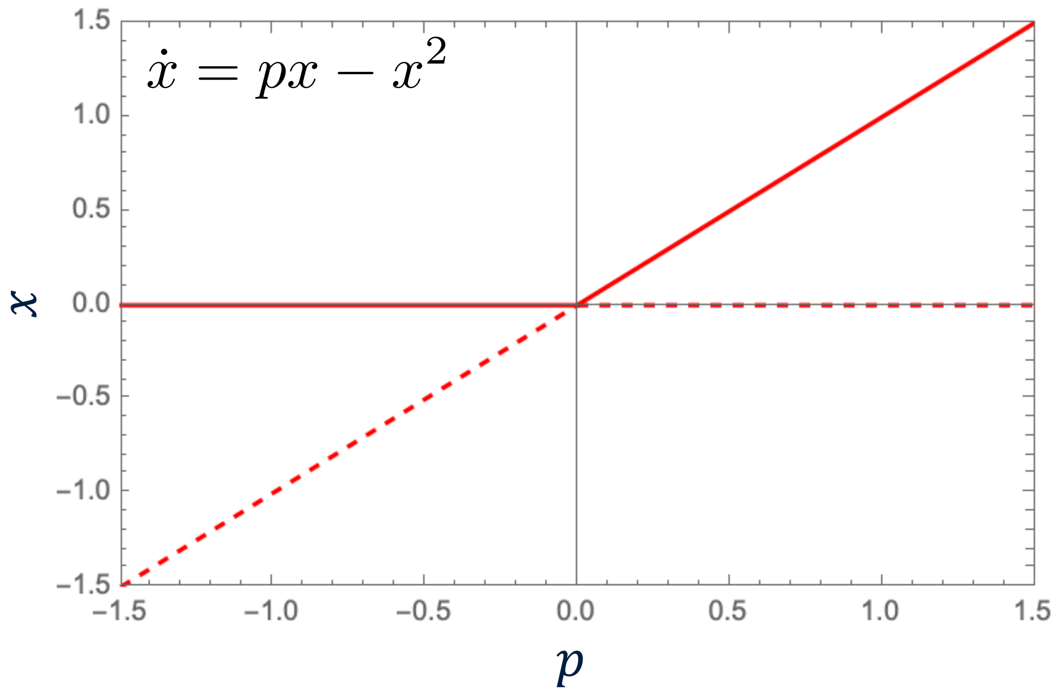
- For a saddle bifurcation we also need f to be **locally linear** in the parameter p and **locally quadratic** in the state x:

$$\left.\frac{\partial f}{\partial p}\right|_{x_0, p_0} \neq 0, \qquad \left.\frac{\partial^2 f}{\partial x^2}\right|_{x_0, p_0} \neq 0$$

Tangency for transcritical bifurcation

- For a bifurcation we need

$$f(x_0, p_0) = 0$$
$$\left.\frac{\partial f}{\partial x}\right|_{x_0,p_0} = 0$$



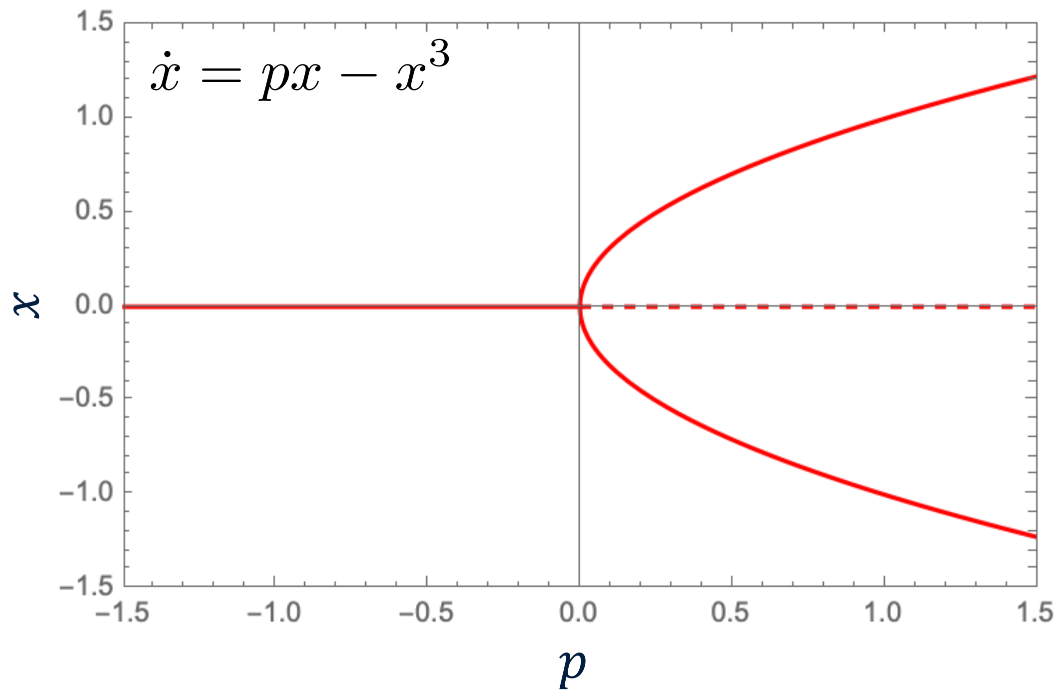
- For a transcritical bifurcation we also need f to be **locally bilinear** in x and p, and **locally quadratic** in the state x

$$\left.\frac{\partial f}{\partial p}\right|_{x_0,p_0} = 0, \quad \left.\frac{\partial^2 f}{\partial x \partial p}\right|_{x_0,p_0} \neq 0, \quad \left.\frac{\partial^2 f}{\partial x^2}\right|_{x_0,p_0} \neq 0$$

Tangency for pitchfork bifurcation

- For a bifurcation we need

$$f(x_0, p_0) = 0$$
$$\left.\frac{\partial f}{\partial x}\right|_{x_0,p_0} = 0$$



- For a pitchfork bifurcation we also need f to be **locally bilinear** in x and p, and **locally cubic** in x

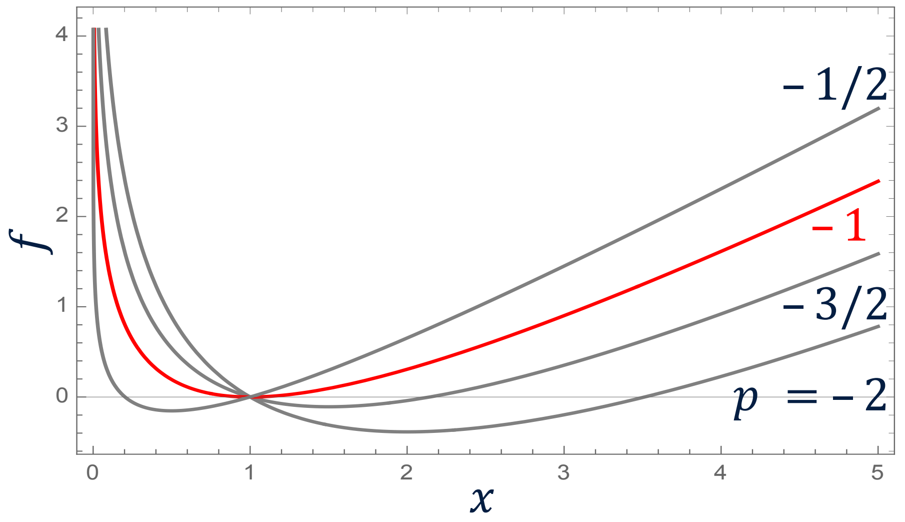
$$\left.\frac{\partial f}{\partial p}\right|_{x_0,p_0} = 0, \quad \left.\frac{\partial^2 f}{\partial x \partial p}\right|_{x_0,p_0} \neq 0, \quad \left.\frac{\partial^2 f}{\partial x^2}\right|_{x_0,p_0} = 0, \quad \left.\frac{\partial^3 f}{\partial x^3}\right|_{x_0,p_0} \neq 0$$

Tangency conditions example

- Consider the system

$$\dot{x} = p \ln x + x - 1, \quad p < 0$$

- This has an equilibrium at $x = 1$
  and has a second equilibrium only if $p \neq -1$



- Tangency conditions show that $p_0 = -1$ is a **bifurcation**:

$$\left.\begin{aligned} p_0 \ln x_0 + x_0 - 1 &= 0 \\ \frac{\partial}{\partial x}(p \ln x + x)\Big|_{x_0, p_0} &= 0 \end{aligned}\right. \implies (x_0, p_0) = (1, -1)$$

- $$\frac{\partial}{\partial p}(p \ln x + x)\Big|_{x_0,p_0} = \ln(x_0) = 0 \qquad \frac{\partial^2}{\partial x \partial p}(p \ln x + x)\Big|_{x_0,p_0} = \frac{1}{x_0} = 1$$

$$\frac{\partial^2}{\partial x^2}(p \ln x + x)\Big|_{x_0,p_0} = -\frac{p_0}{x_0^2} = 1 \qquad \implies \textbf{transcritical}$$

2-D bifurcations

- We can also characterize bifurcations for autonomous systems of higher order (than 1st order), so long as they have just a single scalar parameter p
- If a 2-D system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}; p)$ has an equilibrium at $\mathbf{x}_0$, then bifurcations can be characterized using Sotomayor's theorem (see Perko Sec. 4.2), which uses the Jacobian of $\mathbf{f}$ to formulate higher-dimensional tangency conditions
- We will not apply the theorem in detail here; instead we will explore an example of a 2-D system with a bifurcation

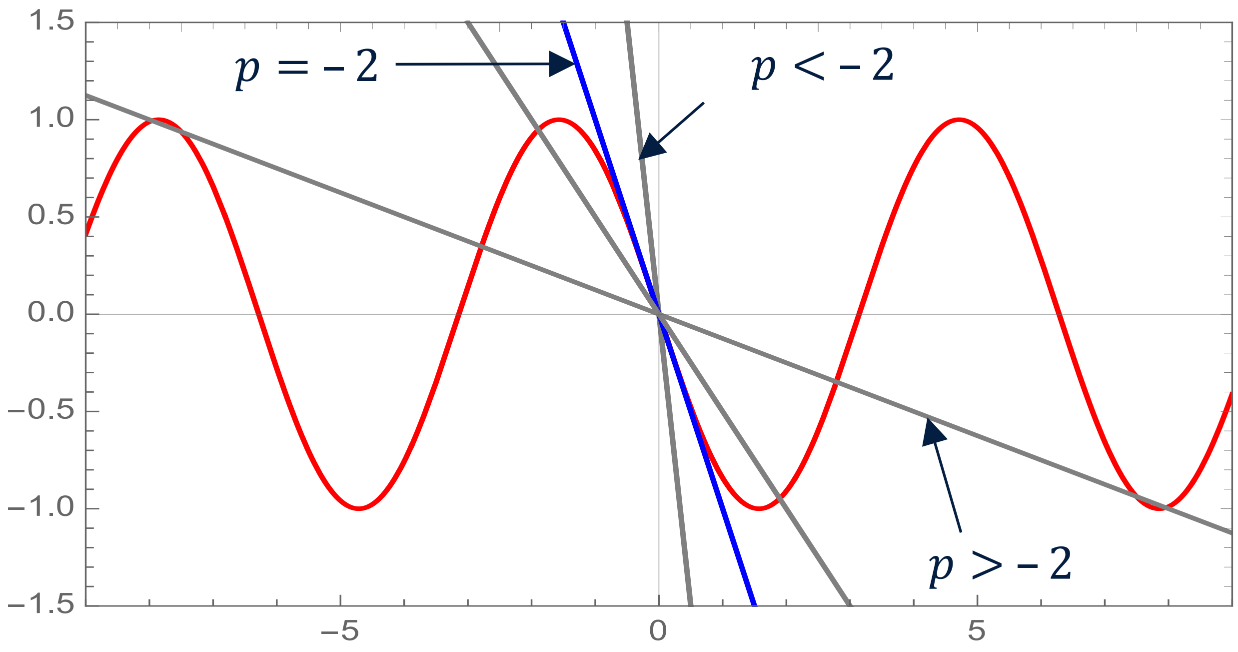
2-D bifurcation example

Does the origin have a bifurcation for the following system?

$$\dot{x} = px + y + \sin x$$

$$\dot{y} = x - y$$

- Find equilibria: $0 = px_0 + y_0 + \sin x_0$, $0 = x_0 - y_0$ ⇒ $y_0 = x_0$, $(p+1)x_0 = -\sin x_0$



if $p = -2$, the line drawn by the left side of this equation is tangent to the function on the right;

expect bifurcation at $p = -2$;

2-D bifurcation example

System $\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \begin{bmatrix}px + y + \sin x\\ x - y\end{bmatrix} = \mathbf{f}(x, y)$

- Jacobian: $D\mathbf{f}(x, y) = \begin{bmatrix}p + \cos x & 1\\ 1 & -1\end{bmatrix}$

  $D\mathbf{f}(0, 0) = \begin{bmatrix}p + 1 & 1\\ 1 & -1\end{bmatrix}$

  eigenvalues: $\text{eig}\big(D\mathbf{f}(0, 0)\big) = \frac{1}{2}\big(p \pm \sqrt{(p + 2)^2 + 4}\big)$

- eigenvalues are: negative if $p < -2$

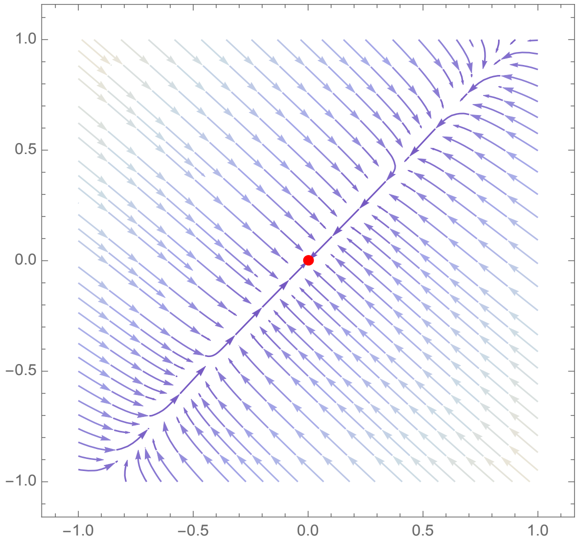
  opposite in sign if $p > -2$ (unstable direction is $\begin{bmatrix}1\\ 1\end{bmatrix}$)

- more than one equilibrium point exists for $p > -2$

2-D bifurcation example

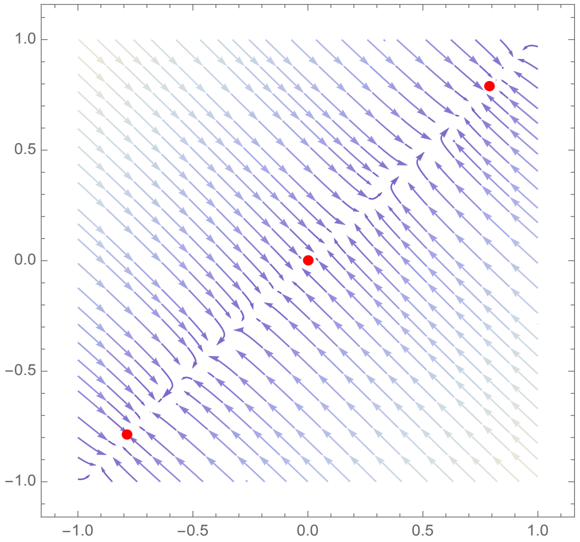
Phase portraits for $p < -2$ and $p > -2$:

$p = -2.1$



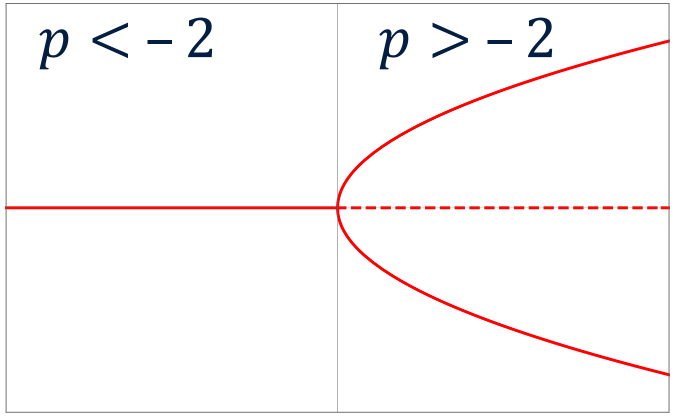
Origin is stable

$p = -1.9$



Origin is a saddle

A **pitchfork** bifurcation:



Hopf bifurcations

- The 2nd order example just considered has an equilibrium point at 0 with:
  - one negative eigenvalue for all values of the parameter p,
  - another eigenvalue passing through 0 at $p = -2$

  The non-hyperbolic behaviour at $p = -2$ is due to a pitchfork bifurcation

- A 2-D system undergoes a **Hopf bifurcation** if the non-hyperbolic point is a centre, i.e. with imaginary eigenvalues

  In this case the real parts of both eigenvalues can change sign

Conditions for a Hopf bifurcation

- Assume a two-dimensional system with a scalar parameter p and equilibrium point $\mathbf{x}^* = \mathbf{x}^*(p)$:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, p), \quad \mathbf{f}(\mathbf{x}^*, p) = 0$$

- The system undergoes a Hopf bifurcation if

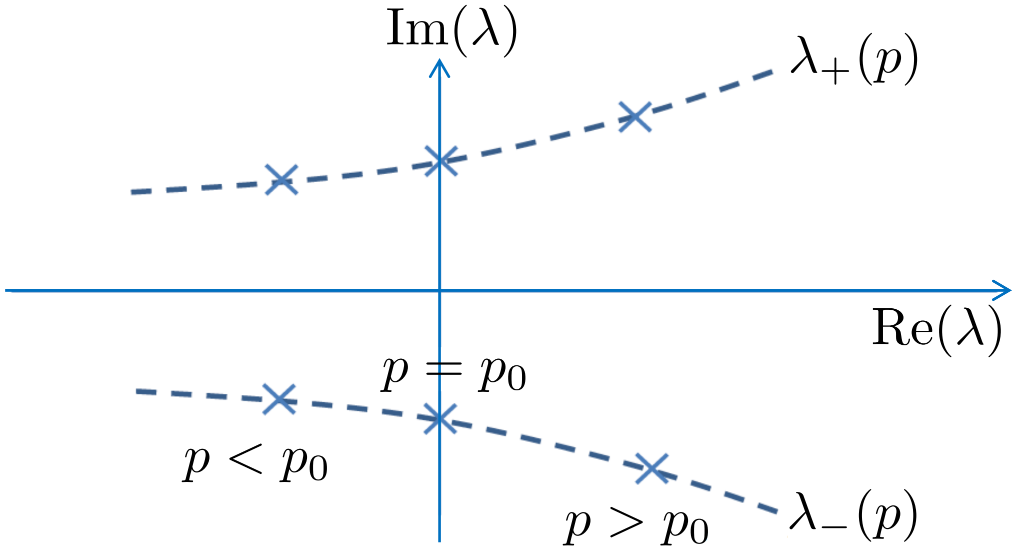
$$\text{eig}\big(D\mathbf{f}(\mathbf{x}^*, p)\big) = \lambda_{\pm}(p) = \alpha(p) \pm j\omega(p)$$

for p in the range

$$p_0 - \epsilon < p < p_0 + \epsilon$$

for some $\epsilon > 0$ with

$$\alpha(p) \begin{cases} < 0 & \text{for } p < p_0 \\ = 0 & \text{for } p = p_0 \\ > 0 & \text{for } p > p_0 \end{cases}$$

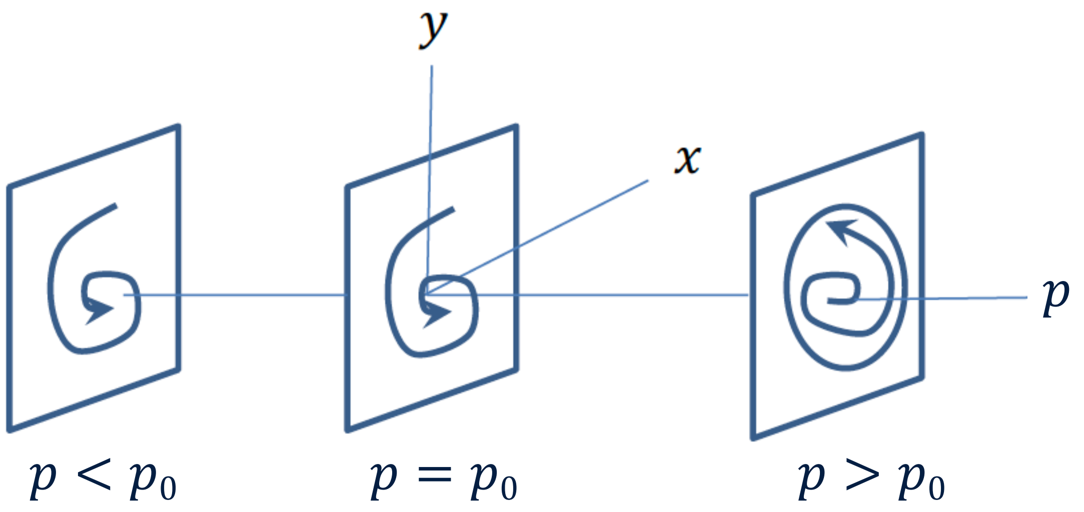


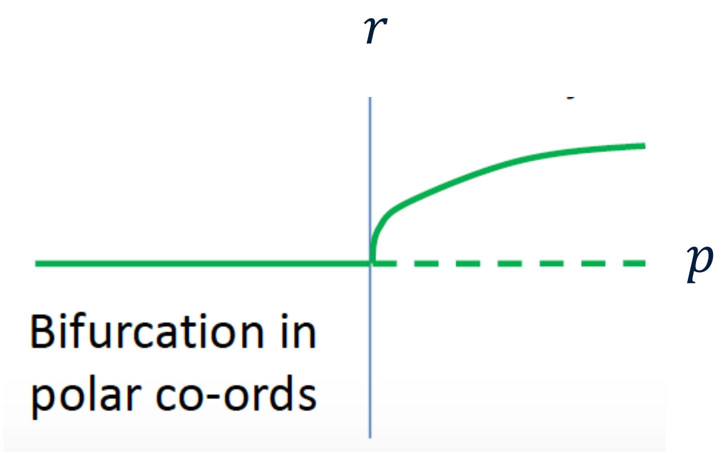
Supercritical Hopf bifurcation

The supercritical Hopf bifurcation is best thought of in polar coordinates (r, θ):

- below the critical value of the parameter, there is a stable spiral equilibrium
- above the critical value, there is an unstable spiral with an enclosing stable limit cycle

the limit cycle's radius r expands with increasing p

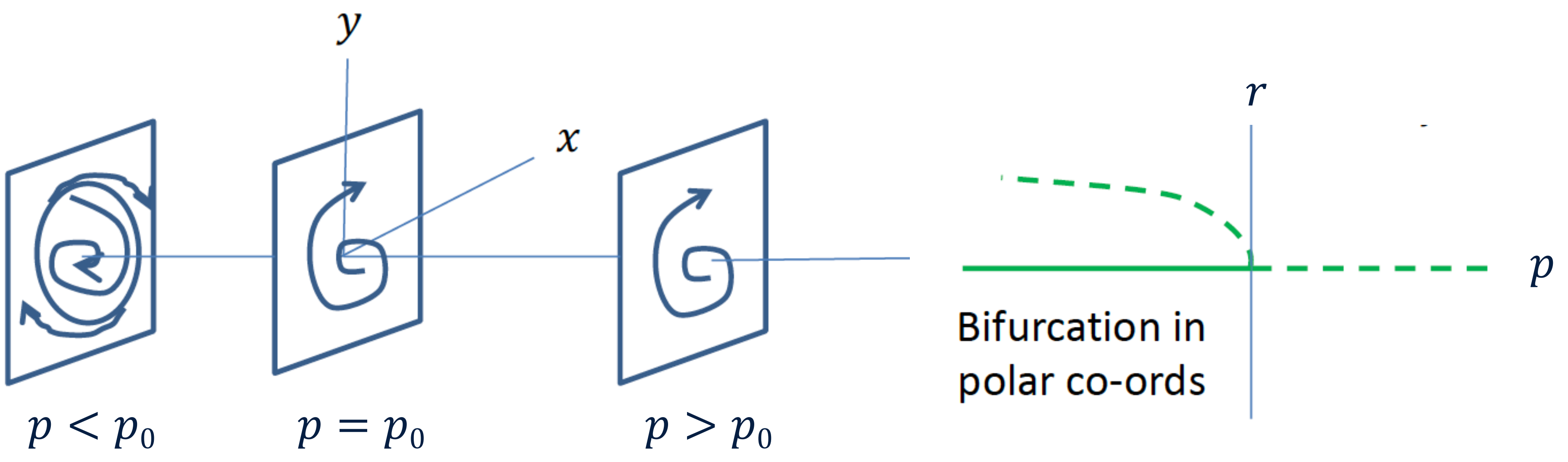




Subcritical Hopf bifurcation

The subcritical Hopf bifurcation behaves as follows:

- below the critical value of p there is a stable spiral surrounded by an unstable limit cycle
- the limit cycle radius shrinks as p increases
- at the critical value the cycle collapses to a fixed point
- above the critical value there is an unstable spiral



Degenerate Hopf bifurcation

The degenerate Hopf bifurcation behaves as follows:

- below the critical value of p there is a stable spiral
- at the critical value of p the spiral becomes a **nonlinear centre** whose orbit is not isolated ($r(t)$ depends on initial conditions)
- above the critical value of p there is an unstable spiral

- Called a **degenerate** bifurcation because there is a non-isolated orbit at the critical parameter value

- The degenerate Hopf bifurcation has no limit cycles for any value of the parameter p

Hopf bifurcation example

Consider the system $\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} px - y + xy^2 \\ x + py + y^3 \end{bmatrix} = \mathbf{f}(x, y)$

- Just one equilibrium point: $\mathbf{f}(x, y) = (0, 0) \implies (x, y) = (0, 0)$

- Eigenvalues of Jocabian at $(x, y) = (0, 0)$:

$$D\mathbf{f}(0,0) = \begin{bmatrix} p + y^2 & -1 + 2xy \\ 1 & p + 3y^2 \end{bmatrix}\Bigg|_{(0,0)} = \begin{bmatrix} p & -1 \\ 1 & p \end{bmatrix} \implies \lambda_\pm = p \pm j$$

- From this we expect a Hopf bifurcation at $p = 0$

Hopf bifurcation example

Transform into polar coordinates: $\dot{r} = \dfrac{x\dot{x} + y\dot{y}}{r} = r(p + r^2 \sin^2\theta)$

$$\dot{\theta} = \frac{x\dot{y} - y\dot{x}}{r^2} = 1$$

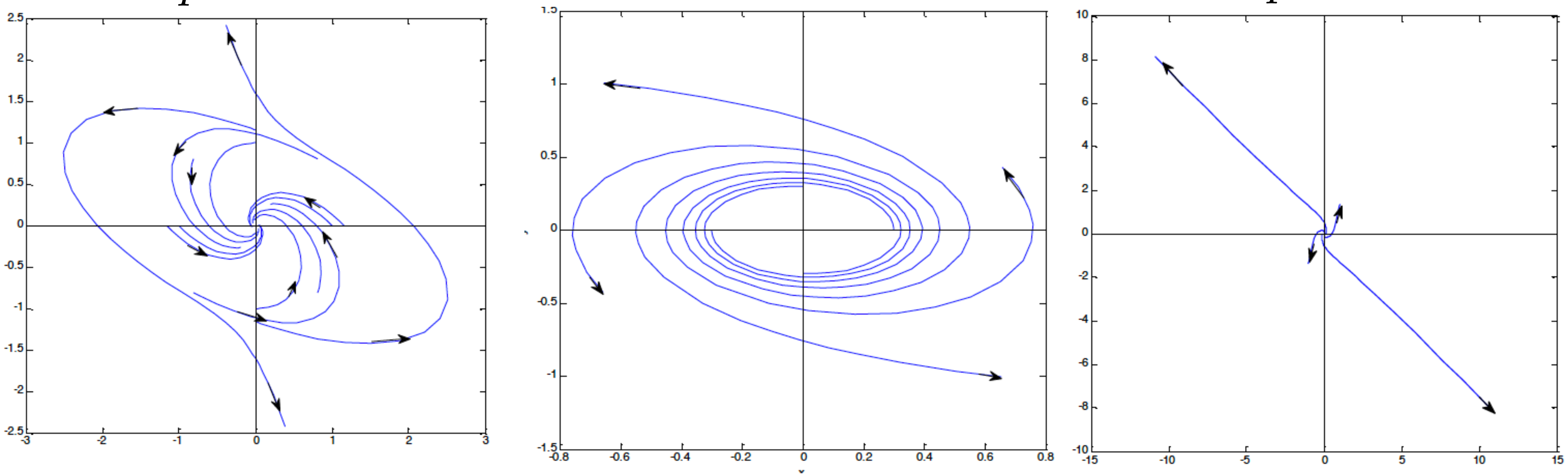
- $p > 0 \implies \dot{r} \geq pr$ — $\dot{r} > 0$ for all t so no limit cycle
- $p = 0 \implies \dot{r} \geq 0$ — $\dot{r} \geq 0$ for all t so no limit cycle
- $p < 0 \implies \dot{r} = pr + ry^2$ — $\dot{r} < 0$ for $y < |p|^{1/2}$ so a stable spiral

Therefore a **subcritical Hopf bifurcation** occurs at $p = 0$,
so expect an unstable limit cycle

Hopf bifurcation example: phase portraits

$$\dot{x} = px - y + xy^2 \qquad \qquad \dot{r} = r(p + r^2 \cos^2 \theta)$$
$$\Longleftrightarrow$$
$$\dot{y} = x + py + y^3 \qquad \qquad \dot{\theta} = 1$$

$p = -1$ $\qquad$ $p = 0$ $\qquad$ $p = 1$



Stable spiral with unstable limit cycle $\qquad$ Unstable spiral $\qquad$ Unstable spiral

Questions?